

Data Abstraction

So far, we have encountered several data types: numbers, strings, booleans, lists, tuples, and dictionaries. But what if we want to represent more complicated things — such as the objects we encounter in real life?

Enter the idea of *data abstraction*, which allows us to build and use objects in code that represent a compound value.

An *abstract data type* consists of two types of functions:

- **Constructors:** functions that build instances of the abstract data type.
- **Selectors:** functions that retrieve information from instances of the ADT.

An integral part of data abstraction is the concept of *abstraction*. Users of the abstract data type have access to an *interface* of defined actions (i.e. constructors and selectors) they can take with the ADT. The user of the ADT does *not* need to know how the constructors and selectors are implemented. The nature of *abstraction* allows whoever uses them to assume that the functions have been written correctly and work as described.

This is called the **abstraction barrier**!

In fact, interacting within an ADT outside of its interface of constructors and selectors is called “violating the abstraction barrier” and is strongly discouraged (even when it doesn’t produce an error).

In this way, data abstraction mimics how we think about the world. For example, a car is a complicated machine, but it has a simple interface by which we interact with it: the wheel, the gas pedal, the gear shift. If you want to drive a car, you don’t need to know how the engine was built or what kind of material the tires are made of. It’s easy to drive a car because of the suppression of complexity that abstraction provides us.

For the car, the principle of the abstraction barrier also applies. If we want to start the car, we should use the provided interface of turning the ignition key, rather than getting out, popping the hood, and messing around with the engine. While you could likely do that if you had enough knowledge, it is unnecessarily costly and highly dependent on how exactly the car was constructed. What are you going to do if the car is electric?

When you’re writing code using ADTs that have been provided for you, make sure that you’re never assuming anything about the ADT other than that it can be constructed with its constructor and its information can be accessed by selectors. Your code should never depend on the specific implementation of the ADT under the hood.

Rationals

Q1: Extending Rationals

First, fill in the following code to implement the rational ADT from lecture.

```

from math import gcd
def make_rat(num, den):
    """Creates a rational number, given a numerator and denominator.

    >>> a = make_rat(2, 4)  2/4 → 1/2
    >>> numer(a)
    1
    >>> denom(a)         gcd(2, 4) → 2
    2
    """
    """*** YOUR CODE HERE ***"""
    g = gcd(num, den)
    return [num // g, den // g]

def numer(rat):
    """Extracts the numerator from a rational number."""
    """*** YOUR CODE HERE ***"""
    return rat[0]

def denom(rat):
    """Extracts the denominator from a rational number."""
    """*** YOUR CODE HERE ***"""
    return rat[1]

# You can use more space on the back if you want

```

Handwritten annotations:

- A circle around `gcd` in the import statement, with an arrow pointing to a box containing $\frac{\text{numerator}}{\text{denominator}}$.
- Handwritten examples: $\frac{2}{4} \rightarrow \frac{1}{2}$, $\text{gcd}(2, 4) \rightarrow 2$, $4 / 2 \rightarrow 2.0$, and $4 // 2 \rightarrow 2$.
- A circled '1' next to the `numer(a)` output.

Q2: Divide

Next, we'll be implementing two additional functions to handle operations between rational numbers.

First, implement `div_rat(x,y)`, which returns the result of dividing rational number `x` by rational number `y`.

```
def div_rat(x, y):
```

```
    """The quotient of rationals x/y.
```

```
    >>> a, b = make_rat(3, 4), make_rat(5, 3)
```

```
    >>> c = div_rat(a, b)
```

```
    >>> numer(c)
```

```
    9
```

```
    >>> denom(c)
```

```
    20
```

```
    """
```

```
    """ YOUR CODE HERE """
```

$$x = \frac{a}{b}, \quad y = \frac{c}{d}$$

$$\frac{x}{y} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

$$\frac{\frac{3}{4}}{\frac{5}{3}} = \frac{3 \times 3}{4 \times 5} = \frac{9}{20}$$

```
    return make_rat ( numer(x) * denom(y),
                      denom(x) * numer(y) )
```

Q3: Less Than

Finally, implement `lt_rat(x, y)`, which returns `True` if and only if rational number `x` is less than rational number `y`.

```
def lt_rat(x, y):
```

```
    """Returns True iff x < y as rational numbers; else False.
```

```
    >>> a, b = make_rat(6, 7), make_rat(12, 16)
```

```
    >>> lt_rat(a, b)
```

```
    False
```

```
    >>> lt_rat(b, a)
```

```
    True
```

```
    >>> lt_rat(a, b)
```

```
    False
```

```
    >>> a, b = make_rat(-6, 7), make_rat(-12, 16)
```

```
    >>> lt_rat(a, b)
```

```
    True
```

```
    >>> lt_rat(b, a)
```

```
    False
```

```
    >>> lt_rat(a, a)
```

```
    False
```

```
    """
```

```
    """ YOUR CODE HERE """
```

$$x = \frac{a}{b}, \quad y = \frac{c}{d}$$

$$x < y \Leftrightarrow \frac{a}{b} < \frac{c}{d}$$

$$\Leftrightarrow \underline{a * d < b * c}$$

if `numer(x) / denom(x) < numer(y) / denom(y)`:

return `True`

else:

return `False`

`return numer(x) / denom(x) < numer(y) / denom(y)`

Trees

In computer science, **trees** are recursive data structures that are widely used in various settings and can be implemented in many ways.

In this class, we consider a **tree** an *abstract data type* (ADT). This will act as our own custom type, much like an integer or string!

A tree has a root **label** and a sequence of **branches**, each of which is also a tree. A **leaf** is a tree with no branches.

- The arguments to the constructor **tree** are the label for the root node and an optional list of branches.
- If no branches parameter is provided, the default value `[]` is used.
- The selectors for these are **label** and **branches**.

Remember **branches** returns a list of trees and not a tree directly. It's important to distinguish between working with a tree and working with a **sequence of** trees.

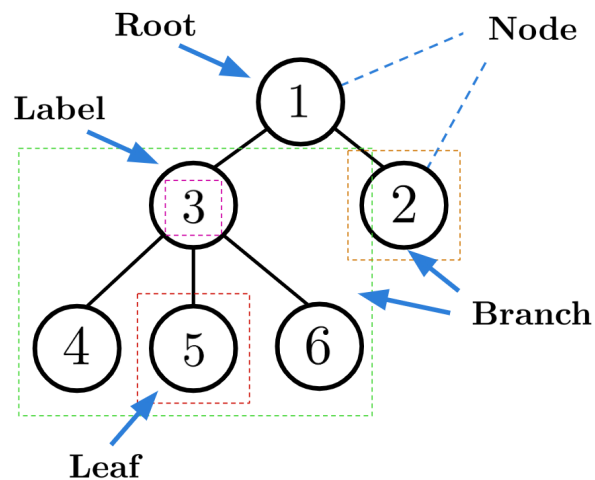
We have also provided a convenience predicate function, **is_leaf**, which returns whether a tree is a leaf.

Let's try to create the tree **t** from above:

```
t2 = tree(3,
        [tree(4),
         tree(5),
         tree(6)])

t1 = tree(1,
        [t2,
         tree(2)])
```

The following is an illustration of tree **t1**.



Example Tree

We note that there are many trees in this data structure. For example, `t1` contains the tree `t2`. So we may refer to a tree as a **node** of a larger structure. Thus, `t1` and `t2` are nodes (locations) in the tree `t1`, and `t1` is the **root node** of this tree.

In the tree `t1`, `t2` is a **child** of `t1`, and `t1` is the **parent** of `t2` since `t2` is a branch of `t1`.

A **path** in a tree is a sequence of nodes where tree `i+1` is the **child** of tree `i`. We say the path is from the first node in the sequence to the last node. The **height** of a tree is this the maximum length **path** to a **leaf** in the tree, where a **leaf** is a tree with no branches.

The **depth** of a node `n` in a tree `t` is the length of the path from the root node of `t` to `n`. For example, in the tree `t1`, the depth of node labeled 4 is 2.

Tree ADT Implementation

For your reference, we have provided our implementation of trees as a data abstraction. However, as with any data abstraction, we should only concern ourselves with what our functions do rather than their specific implementation!

```
def tree(label, branches=[]):
    """Construct a tree with the given label value and a list of branches."""
    return [label] + list(branches)

def label(tree):
    """Return the label value of a tree."""
    return tree[0]

def branches(tree):
    """Return the list of branches of the given tree."""
    return tree[1:]

def is_leaf(tree):
    """Returns True if the tree's list of branches is empty, and False otherwise."""
    return not branches(tree)
```

Q4: Tree Abstraction Barrier

Consider a tree `t` constructed by calling `tree(1, [tree(2), tree(4)])`. For each of the following expressions, answer these two questions:

- What does the expression evaluate to?
- Does the expression violate any abstraction barriers? If so, write an equivalent expression that does not violate abstraction barriers.

1. `label(t)`
2. `t[0]`
3. `label(branches(t)[0])`
4. `is_leaf(t[1:][1])`
5. `[label(b) for b in branches(t)]`
6. **Challenge:** `branches(tree(5, [t, tree(3)]))[0][0]`

Q5: Height

Write a function that returns the height of a tree. Recall that the height of a tree is the number of non-root nodes in the longest path from the root to a leaf.

```
def height(t):
```

```
    """Return the height of a tree.
```

```
    >>> t = tree(3, [tree(5, [tree(1)]), tree(2)])
```

```
    >>> height(t)
```

```
    2
```

```
    >>> t = tree(3, [tree(1), tree(2, [tree(5, [tree(6)]), tree(1)])])
```

```
    >>> height(t)
```

```
    3
```

```
    """
```

```
    """*** YOUR CODE HERE ***"
```

```
    # base case: if t is a leaf
```

```
    if is_leaf(t):
```

```
        return 0
```

```
    # recursive case
```

```
    # find the largest depth from all branches
```

```
    max_depth = 0
```

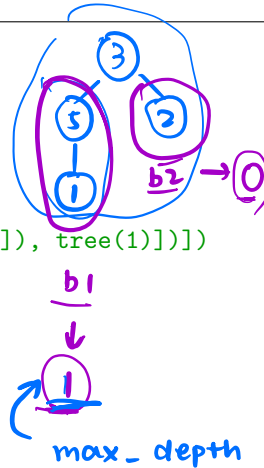
```
    for b in branches(t):
```

```
        max_depth = max(height(b), max_depth)
```

```
    return max_depth + 1
```

```
    max_depth = [height(b) for b in branches(t)]
```

```
# You can use more space on the back if you want
```



Q6: Find Path

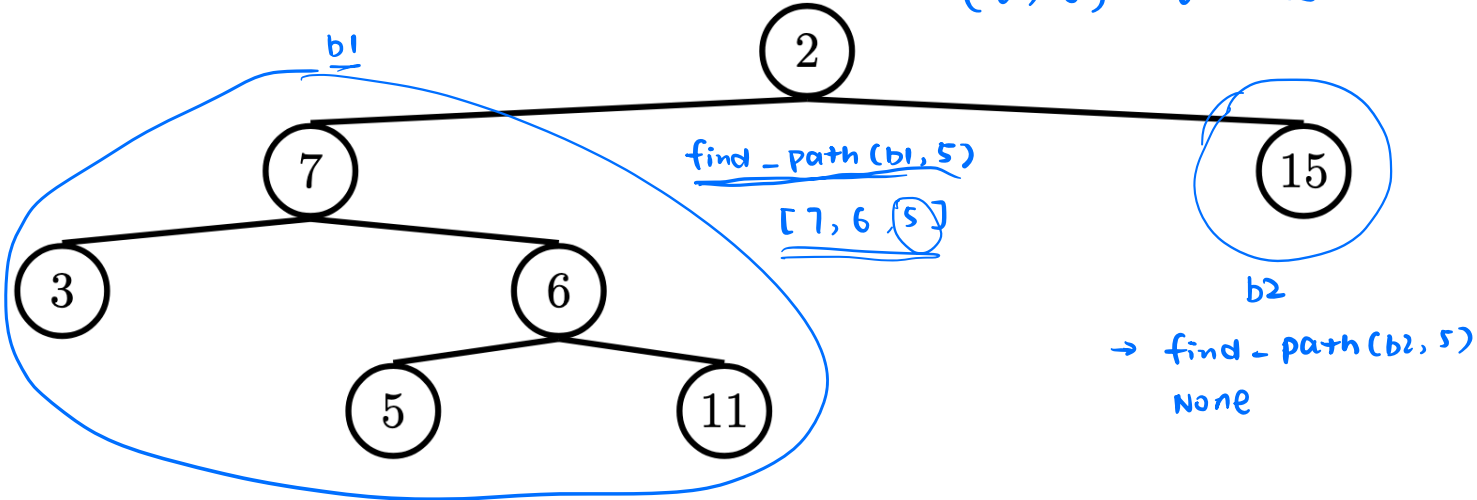
Write a function `find_path` that takes in a tree `t` and a value `x`. It returns a list containing the labels of the nodes along the path from the root of `t` to the node labeled `x`.

If `x` is not a label in `t`, return `None`. Assume that the labels of `t` are unique.

For the following tree, `find_path(t, 5)` should return `[2, 7, 6, 5]`.

$(t, 10) \rightarrow \text{None}$

$(t, 6) \rightarrow [2, 7, 6]$



Example Tree

```
def find_path(t, x):
    """
    >>> t = tree(2, [tree(7, [tree(3), tree(6, [tree(5), tree(11)])]), tree(15)])
    >>> find_path(t, 5)
    [2, 7, 6, 5]
    >>> find_path(t, 10) # returns None
    """
    if label(t) == x:
        return [x]
    for b in branches(t):
        path = find_path(b, x)
        if path != None:
            return [label(t)] + path
```

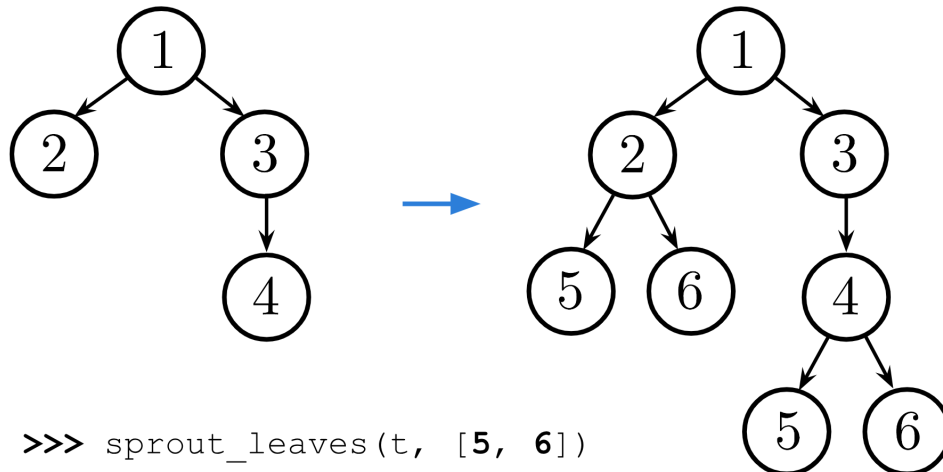
$[2] + [7, 6, 5]$

$2 + []$

Q7: Sprout Leaves

Define a function `sprout_leaves` that takes in a tree, `t`, and a list of leaf labels, `leaves`. It produces a new tree that is identical to `t`, but where each old leaf node has new branches, for each label in `leaves`.

For example, suppose we have the tree `t = tree(1, [tree(2), tree(3, [tree(4)])])`. Calling `sprout_leaves(t, [5, 6])` produces the following tree:



Example Tree


```

def sprout_leaves(t, leaves):
    """Sprout new leaves containing the data in leaves at each leaf in
    the original tree t and return the resulting tree.
    >>> t1 = tree(1, [tree(2), tree(3)])
    >>> print_tree(t1)
    1
     2
     3
    >>> new1 = sprout_leaves(t1, [4, 5])
    >>> print_tree(new1)
    1
     2
      4
      5
     3
      4
      5

    >>> t2 = tree(1, [tree(2, [tree(3)])])
    >>> print_tree(t2)
    1
     2
      3
    >>> new2 = sprout_leaves(t2, [6, 1, 2])
    >>> print_tree(new2)
    1
     2
      3
       6
       1
       2
    """
    "*** YOUR CODE HERE ***"

```

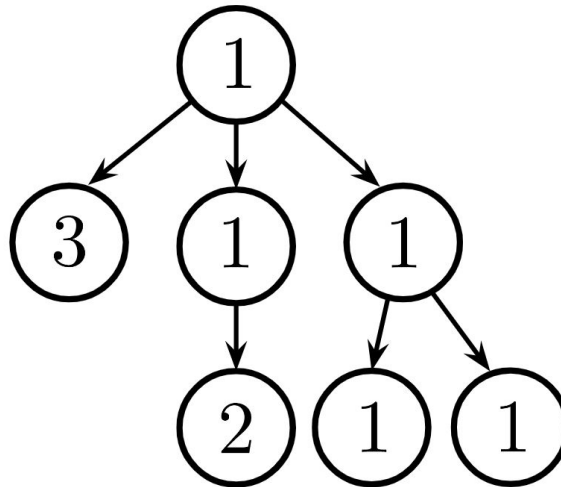
You can use more space on the back if you want

Additional Practice

Q8: Perfectly Balanced

Part A: Implement `sum_tree`, which returns the sum of all the labels in tree `t`.

Part B: Implement `balanced`, which returns whether every branch of `t` has the same total sum and that the branches themselves are also balanced.



Example Tree

- For example, the tree above is balanced because each branch has the same total sum, and each branch is also itself balanced.

Hint: If we ever need to select a specific branch, we will need to break index into our branches list!

Challenge: Solve both of these parts with just 1 line of code each.

```

def sum_tree(t):
    """
    Add all elements in a tree.
    >>> t = tree(4, [tree(2, [tree(3)]), tree(6)])
    >>> sum_tree(t)
    15
    """
    """*** YOUR CODE HERE ***"""

# You can use more space on the back if you want
def balanced(t):
    """
    Checks if each branch has same sum of all elements and
    if each branch is balanced.
    >>> t = tree(1, [tree(3), tree(1, [tree(2)]), tree(1, [tree(1), tree(1)])])
    >>> balanced(t)
    True
    >>> t = tree(1, [t, tree(1)])
    >>> balanced(t)
    False
    >>> t = tree(1, [tree(4), tree(1, [tree(2), tree(1)]), tree(1, [tree(3)])])
    >>> balanced(t)
    False
    """
    """*** YOUR CODE HERE ***"""

# You can use more space on the back if you want

```